Domination Number of Cartesian Product of Graphs

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Let \( G = (V, E) \) be a simple graph with vertex set \( V(G) \) and edge set \( E(G) \). For any vertex \( v \in V(G) \), the neighborhood of \( v \) is the vertex set \( N(v) = \{ u \mid u \text{ is adjacent to } v \} \) and the close neighborhood of \( v \) is the vertex set \( N[v] = N(v) \cup \{v\} \).

- For example, \( N(v_1) = \{v_2, v_5\} \) and \( N[v_1] = \{v_1, v_2, v_5\} \).
Def.

For $D \subseteq V(G)$, $D$ is a *dominating set* of $G$ if $V(G) = \bigcup_{v \in D} N[v]$.

The *domination number* of $G$ is the minimum cardinality of a dominating set of $G$, we denote it by $\gamma(G)$.

For example, $V(C_5) \subseteq N[v_1] \cup N[v_3]$ and it’s impossible to find a vertex to dominate $C_5$, $\gamma(C_5) = 2$. 
Def.

- \( \gamma(G) = \min_{D \subseteq V(G)} \{|D| : V(G) \subseteq \bigcup_{v \in D} N[v]\} \).
- For \( S \) is a subgraph of \( G \),
  \( \gamma_G(S) = \min_{D \subseteq V(G)} \{|D| : V(S) \subseteq \bigcup_{v \in D} N[v]\} \).

Let \( S = G - \{v_4\} - \{v_8\} - \{v_9\} \).
Since \( V(S) \subseteq N[v_6] \cup N[v_5] \),
\( \gamma_G(S) = 2 \).
**Def.**

For two graphs $G$ and $H$, the *Cartesian Product* $G \Box H$ is the graph with vertex set $\{(u, v) | u \in V(G), v \in V(H)\}$ and $(u, v)(u', v') \in E(G \Box H)$ whenever $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

Figure on the below side show a example of $P_4 \Box P_4$ and $P_4 \Box P_4 \Box K_2$. 
In 1963, V.G. Vizing conjectured that for any graph $G$ and $H$,

$$\gamma(G \square H) \geq \gamma(G)\gamma(H).$$
Theorem

(RON, TIBOR, ) If a simple graph $G$ is path, tree, cycle, chordal graph or $\gamma(G) \leq 3$ then for any simple graph $H$

$$\gamma(G \Box H) \geq \gamma(G)\gamma(H).$$
Theorem

(Barcalkin and German)

If $V(G)$ can be covered by $\gamma(G)$ complete subgraphs, then for every graph $H$,

$$\gamma(G \Box H) \geq \gamma(G) \gamma(H).$$
Theorem

(Clark and Suen)

\[ \gamma(G \Box H) \geq \frac{1}{2} \gamma(G) \gamma(H) \]

for any graphs \( G \) and \( H \).
Theorem 1

If a simple graph $G$ has $k$ disjoint complete subgraphs $S_1, S_2, \ldots S_k$ such that $\gamma_G(\bigcup_{i=1}^{k} S_i) = k$, then for any simple graph $H$

$$\gamma(G \Box H) \geq k \gamma(H).$$
**Theorem**

Let $S_1, S_2, \ldots, S_k$ be disjoint sets of $V(G)$. If for any $D \subseteq V(G)$, we have $|D| + |B_D| \geq k$ where $B_D = \{ i \mid S_i \not\subseteq N[D], S_i \cap D = \emptyset \}$, then

$$\gamma(H \Box G) \geq k \gamma(H)$$

for any simple graph $H$.

\[ B_D = \{ 5, 6 \} \]
Def.
A set $X \subseteq V(G)$ is called a 2-packing if $d(u, v) > 2$ for any different vertices $u$ and $v$ of $X$. The 2-packing number $\rho_2(G)$ is the maximum order of a 2-packing of $G$.

Proposition
$\gamma(G) \geq k \geq \rho_2(G)$ for any graph $G$. 
Def.
Let $I$ be an independent set of vertices in the simple graph $G$, the least size of a set of vertices in $G$ that dominates $I$ is denoted by $\gamma_I(G)$, and we denote the largest $\gamma_I(G)$ over all independent sets $I$ in $G$ by $\gamma^i(G)$.

**Proposition**

$k \geq \gamma^i(G)$ for any graph $G$.

PS. $\gamma(G) = \gamma^i(G)$ for any chordal graph $G$. 
Is it possible that $k = \gamma(G)$ for every graph?
Def.

$G$ is called $\gamma_1$-$k$-critical graph if

$$k = \gamma(G) > \gamma(G + e)$$

for any $e \in E(G^c)$.

$G$ is called $\gamma_2$-$k$-critical graph if

$$k = \gamma(G) < \gamma(G - e)$$

for any $e \in E(G)$.

Theorem

If $G$ is a $\gamma_1$-$k$-critical graph and it has $S_1, S_2, \ldots, S_k$ disjoint sets of $V(G)$ such that for any $D \subseteq V(G)$,

$$|D| + |B_D| \geq k$$

then $S_i$ is complete for all $i$. 
Figure: $\gamma_1$-3-critical graphs
Theorem

Let \( D = \{ v_1, v_2, \ldots, v_k \} \) be a dominating set of a \( \gamma_2 \)-k-critical graph \( G \), then for any \( k \) disjoint complete subgraphs \( G_1, G_2, \ldots, G_k \) with \( v_i \in V(G_i) \) for all \( i \), \( \gamma_G(\bigcup_{i=1}^{k} G_i) = k \).
Every $\gamma_1$-k-critical graph can be constructed by adding edges to some $\gamma_2$-k-critical graphs.

\[ K_n \]

\[ \ldots \]

\[ \gamma_2$-(k-1)$-critical graph \]

\[ \gamma_1$-k-critical graph \]

\[ \gamma_2$-k-critical graph \]

Is it possible we can add edges to any $\gamma_1$-k-critical graph to construct a $\gamma_2$-(k-1)-critical graph?
Critical graphs

$\gamma_1 - V(G)$-critical
Critical graphs

\[ \gamma_1 \text{-7-critical} \]
\( \gamma_{1}-6\text{-critical} \)
Critical graphs

$\gamma_1$-5-critical
Critical graphs

$\gamma_1$-4-critical
Critical graphs

$\gamma_1$-3-critical
Critical graphs

$\gamma_1$-2-critical
Critical graphs

$K_n$
Thank you.